Spectral Coherence Polarisation Imaging

Oliver P. Ford

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Abstract

1 Stokes vector and Standard Muller Matricies

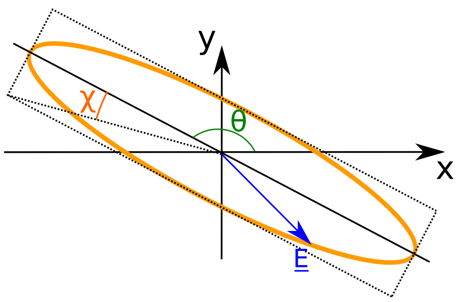


Figure 1: Definition of wave polarisation in terms of ellipticity angle χ and principal polarisation angle θ.

For light with intensity I_0, polarisation angle θ and ellipticity angle χ, stokes vector is:

$$\begin{pmatrix} s_0 \\ s_1 \\ s_2 \\ s_3 \end{pmatrix} = \begin{pmatrix} I_0 \\ I_0 \cos 2\theta \cos 2\chi \\ I_0 \sin 2\theta \cos 2\chi \\ I_0 \sin 2\chi \end{pmatrix} \tag{1}$$

Muller matrix[1] for an ideal polariser at angle θ.

$$\frac{1}{2} \begin{pmatrix} 1 & \cos 2\theta & \sin 2\theta & 0 \\ \cos 2\theta & \cos^2 2\theta & \sin 2\theta \cos 2\theta & 0 \\ \sin 2\theta & \cos 2\theta \sin 2\theta 0 & \sin^2 2\theta & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \tag{2}$$

Polariser at 0^o:

$$\frac{1}{2} \begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \tag{3}$$

Polariser at 45^o:

$$\frac{1}{2}\begin{pmatrix}1 & 0 & 1 & 0\\ 0 & 0 & 0 & 0\\ 1 & 0 & 1 & 0\\ 0 & 0 & 0 & 0\end{pmatrix} \tag{4}$$

Delay plate with phase delay $\Delta\phi$ at angle θ:

$$\begin{pmatrix}1 & 0 & 0 & 0\\ 0 & \cos^2 2\theta + \cos\Delta\phi\sin^2 2\theta & (1-\cos\Delta\phi)\sin 2\theta\cos 2\theta & -\sin\Delta\phi\sin 2\theta\\ 0 & (1-\cos\Delta\phi)\sin 2\theta\cos 2\theta & \sin^2 2\theta + \cos\Delta\phi\cos^2 2\theta & \sin\Delta\phi\cos 2\theta\\ 0 & \sin\Delta\phi\sin 2\theta & -\sin\Delta\phi\cos 2\theta & \cos\Delta\phi\end{pmatrix} \tag{5}$$

Delay plate at 0^o:

$$\begin{pmatrix}1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & \cos\Delta\phi & \sin\Delta\phi\\ 0 & 0 & -\sin\Delta\phi & \cos\Delta\phi\end{pmatrix} \tag{6}$$

Delay plate at 45^o:

$$\begin{pmatrix}1 & 0 & 0 & 0\\ 0 & \cos\Delta\phi & 0 & -\sin\Delta\phi\\ 0 & 0 & 1 & 0\\ 0 & \sin\Delta\phi & 0 & \cos\Delta\phi\end{pmatrix} \tag{7}$$

Delay plate at -45^o:

$$\begin{pmatrix}1 & 0 & 0 & 0\\ 0 & \cos\Delta\phi & 0 & \sin\Delta\phi\\ 0 & 0 & 1 & 0\\ 0 & -\sin\Delta\phi & 0 & \cos\Delta\phi\end{pmatrix} \tag{8}$$

2 Double Spatial Heterodyne

This is the first of J.Howard's Coherence Imaging MSE systems [2]. It encodes the polarisation properties in the difference between two spatial fringe patterns. The polarisation angle θ is of primary interest and in the simplest set-up is encoded in the fringe amplitude difference, giving the system the name 'Amplitude Double Spatial Heterodyne' (ADSH).

The ADSH system consists of:

1. Savart plate at 45^o with phase delay $\Delta\phi_1$
2. Displacer and Delay plates at 0^o with total phase delay $\Delta\phi_2$
3. Polariser at 45^o

Compiling Müller matricies:

$$\underline{S} = \frac{1}{2}\begin{pmatrix}1 & 0 & 1 & 0\\ 0 & 0 & 0 & 0\\ 1 & 0 & 1 & 0\\ 0 & 0 & 0 & 0\end{pmatrix} \cdot \begin{pmatrix}1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & \cos\Delta\phi_2 & \sin\Delta\phi_2\\ 0 & 0 & -\sin\Delta\phi_2 & \cos\Delta\phi_2\end{pmatrix} \cdot \begin{pmatrix}1 & 0 & 0 & 0\\ 0 & \cos\Delta\phi_1 & 0 & -\sin\Delta\phi_1\\ 0 & 0 & 1 & 0\\ 0 & \sin\Delta\phi_1 & 0 & \cos\Delta\phi_1\end{pmatrix} \cdot \begin{pmatrix}s_0\\ s_1\\ s_2\\ s_3\end{pmatrix} \tag{9}$$

$$\underline{S} = \frac{1}{2}\begin{pmatrix}1&0&1&0\\0&0&0&0\\1&0&1&0\\0&0&0&0\end{pmatrix}\cdot\begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&\cos\Delta\phi_2&\sin\Delta\phi_2\\0&0&-\sin\Delta\phi_2&\cos\Delta\phi_2\end{pmatrix}\cdot\begin{pmatrix}s_0\\s_1\cos\Delta\phi_1 - s_3\sin\Delta\phi_1\\s_2\\s_1\sin\Delta\phi_1 + s_3\cos\Delta\phi_1\end{pmatrix} \tag{10}$$

$$\underline{S} = \frac{1}{2}\begin{pmatrix}1&0&1&0\\0&0&0&0\\1&0&1&0\\0&0&0&0\end{pmatrix}\cdot\begin{pmatrix}s_0\\s_1\cos\Delta\phi_1 - s_3\sin\Delta\phi_1\\s_2\cos\Delta\phi_2 + (s_1\sin\Delta\phi_1 + s_3\cos\Delta\phi_1)\sin\Delta\phi_2\\-s_2\sin\Delta\phi_2 + (s_1\sin\Delta\phi_1 + s_3\cos\Delta\phi_1)\cos\Delta\phi_2\end{pmatrix} \tag{11}$$

$$\underline{S} = \frac{1}{2}\begin{pmatrix}s_0 + s_2\cos\Delta\phi_2 + (s_1\sin\Delta\phi_1 + s_3\cos\Delta\phi_1)\sin\Delta\phi_2\\0\\s_0 + s_2\cos\Delta\phi_2 + (s_1\sin\Delta\phi_1 + s_3\cos\Delta\phi_1)\sin\Delta\phi_2\\0\end{pmatrix} \tag{12}$$

Camera only measures intensity, so the first component gives the image equation:

$$2I = s_0 + s_2\cos\Delta\phi_2 + s_1\sin\Delta\phi_1\sin\Delta\phi_2 + s_3\cos\Delta\phi_1\sin\Delta\phi_2 \tag{13}$$

The full image can be built simply by adding this for each Stokes component, since the emission is usually incoherent.

3 Phase Delays

The form of $\Delta\phi_1$ and $\Delta\phi_2$ can be derived from the expression for OPD in a general waveplate [3].

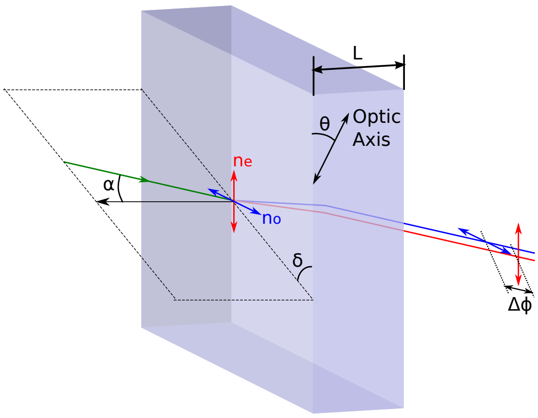


Figure 2: Displacer plate - Homogeneous uniaxial plate with optics axis tilted at angle θ to surface.

L Plate thickness.

λ_0 Wavelength in vacuum.

n_o, n_e Refractive indicies.

θ Angle between optic axis and plate surface.

α Angle of incidence to plate surface.

δ Angle between incidence plane and optic axis plane.

$$\begin{aligned} S &= n_e^2\sin^2\theta + n_o^2\cos^2\theta \\ \frac{\Delta\phi\lambda_0}{2\pi L} &= \begin{pmatrix} \left(n_o^2 - n^2\sin^2\alpha\right)^{\frac{1}{2}} \\ +\frac{n}{S}\left(n_o^2 - n_e^2\right)\sin\theta\cos\theta\cos\delta\sin\alpha + \\ -\frac{n_o}{S}\left[n_e^2 S - \left[n_e^2 - \left(n_e^2 - n_o^2\right)\cos^2\theta\sin^2\delta\right]n^2\sin^2\alpha\right]^{\frac{1}{2}} \end{pmatrix} \end{aligned} \tag{14}$$

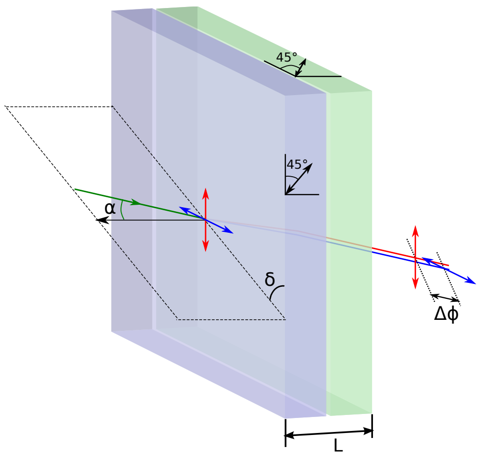
For the IMSE delay plate $\theta = 0^o$:

$$\frac{\Delta\phi\lambda_0}{2\pi L} = \left[n_o - n_e - \sin^2\alpha \frac{1}{2n_o}\left(1 - \frac{n_e}{n_o}\left[1 - \sin^2\delta\left(1 - \frac{n_o^2}{n_e^2}\right)\right]\right)\right] \tag{15}$$

For the displacer plate $\theta = 45^o$:

$$\frac{\Delta\phi\lambda_0}{2\pi L} = \left[\frac{(n_o - n_e)}{2} + \frac{(n_o^2 - n_e^2)}{(n_o^2 + n_e^2)}\cos\delta\sin\alpha\right] \tag{16}$$

Figure 3: Savart plate - Two displacer plates of the same thickness, with the 2nd rotated at 90^o to the first. Ordinary ray in first plate becomes extraordinary in second plate and vice-versa, so that $\delta_2 = \delta_1 - \frac{\pi}{2}$.



2

The savart plate has $\theta = 45^o$ and is two displacer plates with orthogonal δ:

$$\Delta\phi_s = \Delta\phi\left(\tfrac{L}{2}, \delta\right) + \Delta\phi\left(\tfrac{L}{2}, \delta - \tfrac{\pi}{2}\right) \tag{17}$$

$$\Delta\phi_s = \frac{2\pi L}{\sqrt{2}\lambda_0}\frac{(n_o^2 - n_e^2)}{(n_o^2 + n_e^2)}\sin\left(\delta + \tfrac{\pi}{4}\right)\sin\alpha \tag{18}$$

In the OPD equation, δ is in the coordinates of the ordinary and extraordinary axes of the crystal but the Savart is at 45^o in the ADSH system axes, so we can set $\delta \rightarrow (\delta - \frac{\pi}{4})$.

The imaging lens has focal length f_i and is focused at infinty onto the CCD, so in terms of the CCD coords:

$$f_i >> x, y$$
$$\sin\alpha = (x^2 + y^2)/f_i$$
$$\cos\delta = x/(x^2 + y^2)$$
$$\sin\delta = y/(x^2 + y^2)$$

$\Delta\phi_1$ comes from only the Savart plate, and is:

$$\Delta\phi_1 = \frac{\sqrt{2}\pi L_s N}{f_i\lambda_0}y$$
$$= \frac{L_s N}{\sqrt{2}f_i c}\omega y$$
$$\Delta\phi_1 = \alpha\omega y \tag{19}$$

with $N = (n_o^2 - n_e^2)/(n_o^2 + n_e^2)$.

$\Delta\phi_2$ is from the displacer and the delay plate (ignoring $O(\alpha^2$ terms):

$$\begin{aligned}
\Delta\phi_2 &= \frac{2\pi\left(\frac{L_d}{2} + L_w\right)(n_o - n_e)}{\lambda_0} + \frac{2\pi L_d N}{\lambda_0 f_i} x \\
&= \frac{\left(\frac{L_d}{2} + L_w\right)(n_o - n_e)}{c}\omega + \frac{L_d N}{c f_i}\omega x \\
\Delta\phi_2 &= \beta\omega x + \gamma\omega
\end{aligned} \tag{20}$$

For the AUG IMSE:

- Doppler shifted D_α emission: 651nm $< \lambda <$ 655nm so $|\omega'| < 10^{11}$
- All plates are Barium Borate (αBBO): $n_o = 1.666$, $n_e = 1.549$, $N = 0.073$
- Plate Thicknesses: $L_d = 5.4mm$, $L_s = 7.6mm$, $L_w = 1.2mm$, $f_i = 50mm$
- CCD Size: $|x, y| < 5mm$

$$\alpha = \frac{L_s N}{f_i c} \approx 3 \times 10^{-11} \tag{21}$$

$$\beta = \frac{L_d N}{c f_i} \approx 3 \times 10^{-11} \tag{22}$$

$$\gamma = \frac{\left(\frac{L_d}{2} + L_w\right)(n_o - n_e)}{c} \approx 1.5 \times 10^{-12} \tag{23}$$

$$\begin{aligned}
|\alpha\omega y, \beta\omega x| < 10^{-2} &\quad , \quad 1 - \exp(i\alpha\omega y) < 10^{-4} \\
|\gamma\omega| < 10^{-1} &\quad , \quad 1 - \exp(i\gamma\omega) < 10^{-2}
\end{aligned}$$

The image equation, for each individual Stokes component becomes:

$$\begin{aligned}
2I = s_0 \quad & + s_2 \cos(\beta\omega x + \gamma\omega) \\
& + s_1 \sin(\alpha\omega y)\sin(\beta\omega x + \gamma\omega) \\
& + s_3 \cos(\alpha\omega y)\sin(\beta\omega x + \gamma\omega)
\end{aligned} \tag{24}$$

4 Simple MSE multiplet

This section is only for a demonstration of the basic principle of the use of the Delay plate to analyse a net-unpolarised multiplet.

The MSE multiplet consists of the 3 components π^-, σ and π^+. Emission from different components is incoherent as they come from different Quantum events so the image is a superposition of equation 24 for each component. The central σ component has some frequency ω_0 and the π wings have $\omega_0 \pm \Delta\omega$. The polarisation of the πs are aligned at 90^o to the σ, so have $s_1^\pi = -s_1^\sigma$ and $s_2^\pi = -s_2^\sigma$ (Here we choose σ as polarised at θ). For MSE, there is no net circular polarisation so $s_3 = 0$. The intensity of both πs together is the same as the intensity of the polarised fraction of σ.

The s_2 component of the image combined equation becomes:

$$2I_{s2} = s_2 \cos(\beta\omega x + \gamma\omega) \quad (25)$$

$$2I_{s2} = s_2 \begin{pmatrix} \cos(\beta x + \gamma)\omega_0 \\ -\frac{1}{2}\cos(\beta x + \gamma)(\omega_0 + \Delta\omega) \\ -\frac{1}{2}\cos(\beta x + \gamma)(\omega_0 - \Delta\omega) \end{pmatrix} \quad (26)$$

$$2I_{s2} = s_2[1 - \cos(\beta x + \gamma)\Delta\omega]\cos(\beta x + \gamma)\omega_0 \quad (27)$$

For the s_1 component:

$$2I_{s1} = s_1 \sin(\alpha\omega y)\sin(\beta\omega x + \gamma\omega) \quad (28)$$

$$4I_{s1} = s_1[\cos(\beta x + \gamma - \alpha y)\omega - \cos(\beta x + \gamma + \alpha y)\omega] \quad (29)$$

$$8I_{s1} = s_1 \begin{pmatrix} \cos(\beta x + \gamma - \alpha y)\omega_0\,[2 - \cos(\beta x + \gamma - \alpha y)\Delta\omega - \cos(\beta x + \gamma - \alpha y)\Delta\omega] \\ -\cos(\beta x + \gamma + \alpha y)\omega_0\,[2 - \cos(\beta x + \gamma + \alpha y)\Delta\omega - \cos(\beta x + \gamma + \alpha y)\Delta\omega] \end{pmatrix} \quad (30)$$

For AUG MSE, $\Delta\lambda \approx 0.5nm$ for π, σ separation, so:

$$\Delta\omega \approx 2 \times 10^{-12} \quad (31)$$

$$\gamma\Delta\omega \approx 2 \quad (32)$$

$$|\alpha\Delta\omega y| \approx |\beta\Delta\omega x| < 0.3 \quad (33)$$

$$(34)$$

The $\alpha\Delta\omega y$ and $\beta\Delta\omega x$ are relatively small as arguments to the cosine and sine, so we can approximate $\cos(\beta x + \gamma + \alpha y)\Delta\omega \to \cos\gamma\Delta\omega$:

$$4I_{s1} = s_1\,[1 - \cos\gamma\Delta\omega]\left(\cos(\beta x + \gamma - \alpha y)\omega_0 - \cos(\beta x + \gamma + \alpha y)\omega_0\right) \quad (35)$$

Combining I_1 and I_2 back to the full image:

$$2I = s_0 + 2I_{s2} + 2I_{s1} \quad (36)$$

$$= s_0 + [1 - \cos\gamma\Delta\omega] \begin{pmatrix} s_2\cos(\beta x + \gamma)\omega_0 \\ \frac{1}{2}s_1\cos(\beta x + \gamma - \alpha y)\omega_0 \\ -\frac{1}{2}s_1\cos(\beta x + \gamma + \alpha y)\omega_0 \end{pmatrix} \quad (37)$$

$$(38)$$

This is the same as the image equation (equation 24) for the σ component, except with the fringe amplitude is reduced relative to the background intensity. This reduction is the contrast:

$$\zeta = [1 - \cos\gamma\Delta\omega] \quad (39)$$

If there is no fixed delay $\gamma = 0$, then $\zeta = 0$ and there will be no fringes. This is due to the π and σ cancelling out. The contrast is a maximum at $\gamma\Delta\omega = 90^o$ and the wave plate thickness L_w is chosen such that this is true for the expected splitting ($\Delta\omega$) via equation 23. More generally, the constrast function is actually the fourier transform of spectrum.

5 Demodulation (Single Wavelength)

The DSH image equation can be expanded as:

$$\begin{aligned} 2I = s_0 & +s_2 && \cos(\beta\omega x + \gamma\omega) \\ & +\tfrac{1}{2}s_1 && \cos(\alpha\omega y - \beta\omega x - \gamma\omega) \\ & -\tfrac{1}{2}s_1 && \cos(\alpha\omega y + \beta\omega x + \gamma\omega) \\ & +\tfrac{1}{2}s_3 && \sin(\alpha\omega y + \beta\omega x + \gamma\omega) \\ & -\tfrac{1}{2}s_3 && \sin(\alpha\omega y - \beta\omega x - \gamma\omega) \end{aligned}$$

In exponential form:

$$\begin{aligned} 2I = & \quad s_0 \\ & +\tfrac{1}{2}s_2 \exp i(+\beta\omega x + \gamma\omega) && +\tfrac{1}{2}s_2 \exp i(-\beta\omega x - \gamma\omega) \\ & +\tfrac{1}{4}s_1 \exp i(+\alpha\omega y - \beta\omega x - \gamma\omega) && +\tfrac{1}{4}s_1 \exp i(-\alpha\omega y + \beta\omega x + \gamma\omega) \\ & -\tfrac{1}{4}s_1 \exp i(+\alpha\omega y + \beta\omega x + \gamma\omega) && -\tfrac{1}{4}s_1 \exp i(-\alpha\omega y - \beta\omega x - \gamma\omega) \\ & -i\tfrac{1}{4}s_3 \exp i(+\alpha\omega y + \beta\omega x + \gamma\omega) && +i\tfrac{1}{4}s_3 \exp i(-\alpha\omega y - \beta\omega x - \gamma\omega) \\ & +i\tfrac{1}{4}s_3 \exp i(+\alpha\omega y - \beta\omega x - \gamma\omega) && -i\tfrac{1}{4}s_3 \exp i(-\alpha\omega y + \beta\omega x + \gamma\omega) \\ 8I = & \quad 4s_0 \\ & +2s_2 && \exp i(+\beta\omega x + \gamma\omega) \\ & +2s_2 && \exp i(-\beta\omega x - \gamma\omega) \\ & -[s_1 + is_3] && \exp i(+\beta\omega x + \gamma\omega + \alpha\omega y) \\ & +[s_1 + is_3] && \exp i(-\beta\omega x - \gamma\omega + \alpha\omega y) \\ & +[s_1 - is_3] && \exp i(+\beta\omega x + \gamma\omega - \alpha\omega y) \\ & -[s_1 - is_3] && \exp i(-\beta\omega x - \gamma\omega - \alpha\omega y) \end{aligned}$$

The seven components of this, are denoted by their signs in (x, y): $(0, 0), (+, 0), (+, +), (+, -)$ etc.

For a single ω, the 2D Fourier Transform (FT) from (x, y) into some (kx, ky) consists of the FT of the stokes components, convolved with a delta functions at the respective frequency. e.g., the $(+, +)$ component is:

$$FT[8I] = ... \quad - \quad FT[s_1 + is_3] \otimes \delta\,(k_x - \beta\omega)\,\delta\,(k_y - \alpha\omega) \tag{40}$$

If variations in $s_i(x, y)$ are slow, these appear separated in the FT, as in figure 4 and can be filtered (cut out) and individually inverse FT'ed.

$$\begin{aligned} 8I(+,+) &= (s_1 - is_3) \exp i(+\beta\omega x + \gamma\omega + \alpha\omega y) \\ |8I(+,+)| &= |s_1 - is_3| \end{aligned}$$

All 3 components

$$|8I(+,0)| = 2s_2 \tag{41}$$

$$|8I(+,+)| = |s_1 - is_3| \tag{42}$$

$$|8I(+,-)| = |s_1 + is_3| \tag{43}$$

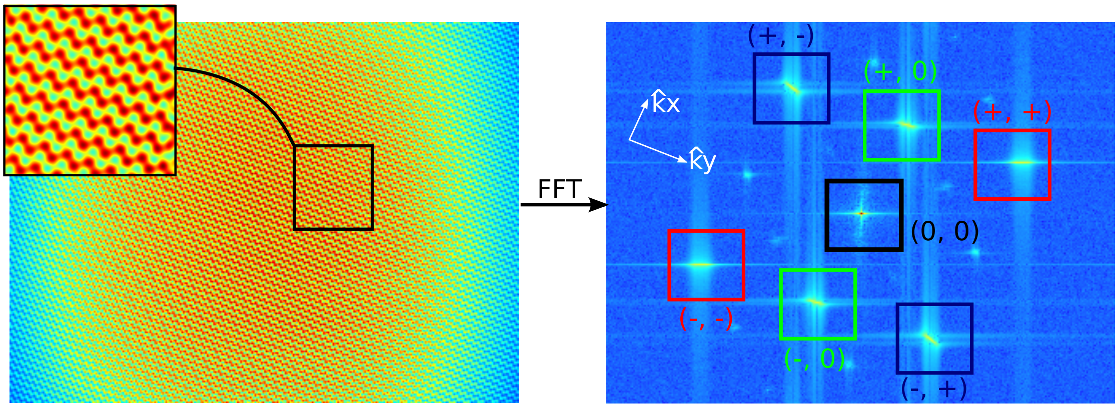


Figure 4: A typical DSH image and Fourier Transform. Typical filters for the 7 principle components are shown. Other visible small components result from small (¡0.3°) misalignments of the plates to the final polariser. Spectral bleeding from edge effects can also be seen.

If $s_3 << s_2$ (small ellipticity χ, or polarisation angle $\psi \approx 45^o$)

$$\frac{|I(+,0)|}{|I(+,+)|} \approx \frac{2s_2}{s_1} = 2\tan 2\theta \tag{44}$$

which is the desired polarisation angle image.

6 Full Spectrum

We really have a continuous spectrum over which the stokes vector changes both in intensity and polarisation state $\underline{s}(\omega)$. Accross the MSE spectrum, the polarisation variation is not large, except for rapid changes $\Delta\theta \approx 90^o$ as it passes from σ dominated to π dominated, which appear as sign changes in the Stokes components. The stokes components can therefore be separated into $\underline{s}(\omega) = I(\omega)\,\underline{s}$, with the sign changes held in $I(\omega)$. The full image is an integral over all frequencies but is best expressed relative to some central ω_0. e.g. for $(+,+)$:

$$8I(x,y) = ... \quad + \quad s_1 \int I(\omega') \exp i\left[(\omega_0 - \omega')(\beta x + \alpha y + \gamma)\right] d\omega' + ... \tag{45}$$

The FT components are now convolved not with delta functions, but with integrals over the spectrum:

$$FT[8I] \quad = \quad ... + FT[s_1] \otimes \int I(\omega') \exp i\gamma(\omega_0 - \omega') \begin{pmatrix} \delta\left(k_x - \beta(\omega_0 - \omega')\right) \\ \times\, \delta\left(k_y - \alpha(\omega_0 - \omega')\right) \end{pmatrix} d\omega' + ... \tag{46}$$

Approximately the same amount of the spectrum is from the σ as from the π components and as these have opposite sign in $I(\omega)$ the integral would vanish. The delay plate, which determines γ should therefore be set such that the sign change of the oscillating $\exp i\gamma(\omega_0 - \omega')$ term roughly cancels the sign change in $I(\omega)$.

The range of ω is small ($\frac{\omega'}{\omega_0} < 0.004$), so the components are still separable. Cutting out just the $(+,+)$ part and applying the inverse FT still gives something oscillating at the central frequency ω_0:

$$8I(+,+) = s_1 \int I(\omega') \exp i\left[(\omega_0 - \omega')(\beta x + \alpha y + \gamma)\right] d\omega' \tag{47}$$

$$= s_1 \left(\int I(\omega') \exp\left[-i\omega'(\beta x + \alpha y + \gamma)\right] d\omega' \right) \exp\left[i\omega_0(\beta x + \alpha y + \gamma)\right] \tag{48}$$

The amplitude of that oscillation is now multipled by a function of the spectrum:

$$|8I(+,+)| = |s_1|\, |\Gamma(\beta x + \alpha y + \gamma)| \tag{49}$$

$$\Gamma(x') = \int I(\omega') \exp\left[-i\omega' x'\right] d\omega' \tag{50}$$

where $\Gamma(x')$ is the amplitude of the 1D FT of the spectrum, with x' as the spatial variable.

The $(+,-)$ component is identical but with $x' = (\beta x - \alpha y + \gamma)$ and the $(+,0)$ component with $x' = (\beta x + \gamma)$.

The problem now is that we get (for $s_3 << 1$):

$$\frac{|I(+,0)|}{|I(+,+)|} = 2\frac{s_2}{s_1} \frac{|\Gamma(\beta x + \gamma)|}{|\Gamma(\beta x + \alpha y + \gamma)|} \tag{51}$$

and need to find a way to extract $\frac{s_2}{s_1}$ from this.

7 Spectrum Resilient demodulation

Returning to the full image equation components:

$$\begin{aligned}
8I(0,0) &= \quad 4s_0 \\
8I(+,0) &= \quad +2s_2 \quad \exp i(+\beta\omega x + \gamma\omega) \\
8I(+,+) &= -[s_1 + is_3] \quad \exp i(+\beta\omega x + \gamma\omega + \alpha\omega y) \\
8I(+,-) &= +[s_1 - is_3] \quad \exp i(+\beta\omega x + \gamma\omega - \alpha\omega y) \\
8I(-,0) &= \quad +2s_2 \quad \exp i(-\beta\omega x - \gamma\omega) \\
8I(-,+) &= +[s_1 + is_3] \quad \exp i(-\beta\omega x - \gamma\omega + \alpha\omega y) \\
8I(-,-) &= -[s_1 - is_3] \quad \exp i(-\beta\omega x - \gamma\omega - \alpha\omega y)
\end{aligned}$$

Try to balance the Γ terms by using $I(+,-)$ instead of the modulus to remove the $exp(i\omega_0...)$ terms:

$$\frac{4I(+,+)I(+,-)}{I(+,0)^2} = \frac{-(s_1^2 + s_3^2)}{+s_2^2} \frac{\Gamma(\beta x + \gamma + \alpha y)\Gamma(\beta x + \gamma - \alpha y)}{\Gamma(\beta x + \gamma)^2} \tag{52}$$

Substituting:

$$\omega_a z = \omega'(\beta x + \alpha y + \gamma) \tag{53}$$

$$\omega' = \frac{\omega_a z}{(\beta x + \alpha y + \gamma)} \tag{54}$$

$$I_a(\omega) = I\left(\frac{z}{(\beta x + \alpha y + \gamma)} \; \omega \right) \tag{55}$$

$$\Gamma(\beta x + \gamma + \alpha y) = \int I(\omega') \exp\left[-i\omega'(\beta x + \alpha y + \gamma)\right] d\omega' \tag{56}$$

$$= \frac{z}{(\beta x + \alpha y + \gamma)} \int I_a(\omega_a) \exp\left[-i\omega_a z\right] d\omega_a \tag{57}$$

And, erm....

$$\frac{4I(+,+)I(+,-)}{I(+,0)^2} = \frac{(\beta x+\gamma)^2}{(\beta x+\alpha y+\gamma)(\beta x-\alpha y+\gamma)} \tag{58}$$

$$\frac{\left(\int I_a(\omega_a)\exp\left[-i\omega_a z\right]d\omega_a\right)\left(\int I_b(\omega_b)\exp\left[-i\omega_b z\right]d\omega_b\right)}{\left(\int I_c(\omega_c)\exp\left[-i\omega_c z\right]d\omega_c\right)^2} \tag{59}$$

This isn't going anywhere, so I haven't actually proved they cancel, but they look.. err... *balanced*, and it works much better in model and in the lab. So, keeping ellipticity this time, it is:

$$\frac{4I(+,+)I(+,-)}{I(+,0)^2} \approx \frac{-(s_1^2+s_3^2)}{+s_2^2} \tag{60}$$

$$= \frac{-(\cos^2 2\theta\cos^2\chi+\sin^2\chi)}{\sin^2 2\theta\cos^2\chi} \tag{61}$$

$$= \frac{-(\cos^2 2\theta+\tan^2\chi)}{\sin^2 2\theta} \tag{62}$$

$$\frac{4I(+,+)I(+,-)}{I(+,0)^2} = -\tan^2 2(\theta+45^o)-\frac{\tan^2\chi}{\sin^2 2\theta} \tag{63}$$

8 Ellipticity

Ellipticity effect is strong if $\sin(2\theta) << 1$, even if it is small, so it is useful to be able to measure it.

Adding a $\lambda/4$ plate before system gives:

$$\begin{pmatrix} s_0' \\ s_1' \\ s_2' \\ s_3' \end{pmatrix} = \begin{pmatrix} s_0 \\ s_1 \\ s_3 \\ -s_2 \end{pmatrix} \tag{64}$$

We do this with an Ferro-electric crystal (FLC). In principle, it is always $\lambda/4$ phase shift, but when off is aligned to the first plate axes so does nothing. More accurately, it swaps s_1 with $-s_3$, which has no effect on the spectrum balanced expression for θ in equation 63. When on, it is a $\lambda/4$ plate.

Amplitude is now:

$$\left[\frac{4I(+,+)I(+,-)}{I(+,0)^2}\right]^{ON} = \frac{-(s_1^2+s_2^2)}{+s_3^2}\,\frac{\Gamma(x+y)\Gamma(x-y)}{\Gamma(x)} \tag{65}$$

$$\approx -\frac{(\cos^2 2\theta\cos^2\chi+\sin^2 2\theta\cos^2\chi)}{\sin^2\chi} \tag{66}$$

$$= -\tan^2(45^o-\chi) \tag{67}$$

$$\tag{68}$$

The reciprocal gives $tan^2(\chi)$, which can now be removed from the original:

$$X = \left[\frac{4I(+,+)I(+,-)}{I(+,0)^2}\right]^{OFF} = \frac{-(\cos^2 2\theta + \tan^2\chi)}{\sin^2 2\theta} \tag{69}$$

$$-X\sin^2 2\theta = \cos^2 2\theta + \tan^2\chi \tag{70}$$

$$(1-X)\sin^2 2\theta = \cos^2 2\theta + \tan^2\chi + \sin^2 2\theta \tag{71}$$

$$\sin^2 2\theta = \frac{\tan^2\chi}{(1-X)} \tag{72}$$

9 Off-spec FLC

The supposedly $\lambda/4$ FLC can actually have phase of $180^o + \delta\phi$ (A.Thor). When switched off, at any $\delta\phi$, the effect is still only to rotate arbitrarily between s_1 and $-s_3$. Any $\delta\phi$ still does not effect the result of equation 63.

When switched on however, the FLCs effect is now:

$$\cos\Delta\phi = \cos(\tfrac{\pi}{2} + \delta\phi) = -\sin\delta\phi \tag{73}$$

$$\sin\Delta\phi = \sin(\tfrac{\pi}{2} + \delta\phi) = \cos\delta\phi \tag{74}$$

$$\begin{pmatrix} s_0' \\ s_1' \\ s_2' \\ s_3' \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -\sin\delta\phi & \cos\delta\phi \\ 0 & 0 & -\cos\delta\phi & -\sin\delta\phi \end{pmatrix} \begin{pmatrix} s_0 \\ s_1 \\ s_2 \\ s_3 \end{pmatrix} \tag{75}$$

The FLC used in the AUG IMSE has $\Delta\phi = 81.419^o$ at $\lambda = 653nm$. At room temperatue, it is also has a non-45° switching angle, but this returns to 45° at a temperature of 47°C. A.Thor's results have a similar $\delta\phi$ of a few degrees, so small angles approximation is ok here.

$$s' \approx \begin{pmatrix} s_0 \\ s_1 \\ s_3 - s_2\delta\phi \\ -s_2 - s_3\delta\phi \end{pmatrix} \tag{76}$$

Define $\alpha = s_2/s_1$, $\beta = s_3/s_1$. Amplitude with FLC off is the same as before:

$$A = \left[\frac{4I(+,+)I(+,-)}{I(+,0)^2}\right]^{OFF} \approx \frac{-(s_1^2 + s_3^2)}{+s_2^2} \tag{77}$$

$$s_3^2 = s_1^2 - As_2^2 \tag{78}$$

$$\beta^2 = 1 - \alpha^2 A \tag{79}$$

Ampltiude with FLC on is now:

$$B = \left[\frac{4I^F(+,+)I^F(+,-)}{I^F(+,0)^2}\right]^{ON} \approx \frac{-(s_1'^2 + s_3'^2)}{+s_2'^2} \quad (80)$$

$$B = -\frac{s_1^2 + s_2^2 + s_3^2\delta\phi^2 + 2s_2 s_3\delta\phi}{s_3^2 + s_2^2\delta\phi^2 - 2s_2 s_3\delta\phi} \quad (81)$$

$$B = -\frac{1 + \alpha^2 + \beta^2\delta\phi^2 + 2\alpha\beta\delta\phi}{\beta^2 + \alpha^2\delta\phi^2 - 2\alpha\beta\delta\phi} \quad (82)$$

$$2\alpha\beta B\delta\phi - 2\alpha\beta\delta\phi = 1 + B\beta^2 + \alpha^2 B\delta\phi^2 + \alpha^2 + \beta^2\delta\phi^2 \quad (83)$$

$$2\alpha\beta\delta\phi(B-1) = 1 + \beta^2(B + \delta\phi^2) + \alpha^2(1 + B\delta\phi^2) \quad (84)$$

$$(85)$$

Square and substitude β^2 from FLC off ampltude:

$$(2\delta\phi(B-1))^2\alpha^2\beta^2 = \left[1 + \beta^2(B + \delta\phi^2) + \alpha^2(1 + B\delta\phi^2)\right]^2 \quad (86)$$

$$= \left[1 + (1 - \alpha^2 A)(B + \delta\phi^2) + \alpha^2(1 + B\delta\phi^2)\right]^2 \quad (87)$$

$$= \left[1 + (B + \delta\phi^2) - \alpha^2 A(B + \delta\phi^2) + \alpha^2(1 + B\delta\phi^2)\right]^2 \quad (88)$$

$$= \left[(1 + B + \delta\phi^2) + \alpha^2(1 + B\delta\phi^2 - AB - A\delta\phi^2)\right]^2 \quad (89)$$

$$(90)$$

Collecting up some constants:

$$X = (1 + B\delta\phi^2 - AB - A\delta\phi^2) \quad (91)$$

$$Y = 2\delta\phi(B-1) \quad (92)$$

$$Z = (1 + B + \delta\phi^2) \quad (93)$$

$$Y^2\alpha^2\beta^2 = \left[\alpha^2 X + Z\right]^2 \quad (94)$$

$$(95)$$

and substituting the LHS β^2:

$$Y^2\alpha^2(1 - \alpha^2 A) = \alpha^4 X^2 + Z^2 + 2\alpha^2 XZ \quad (96)$$

$$0 = -(AY^2 + X^2)\alpha^4 + (Y^2 - 2XZ)\alpha^2 - Z^2 \quad (97)$$

$$(98)$$

Solving the quadratic for the polarisation:

$$\tan^2(2\theta) = \alpha^2 \quad (99)$$

$$= \frac{(Y^2 - 2XZ) \pm \sqrt{(Y^2 - 2XZ)^2 - 4(AY^2 + X^2)Z^2}}{2(AY^2 + X^2)} \quad (100)$$

And recover χ for each solution via A:

$$\beta^2 = 1 - \alpha^2 A \quad (101)$$

$$\frac{\tan^2(2\chi)}{\cos^2(2\theta)} = 1 - \tan^2(2\theta)A \quad (102)$$

$$\tan^2(2\chi) = \cos^2(2\theta) - A\sin^2(2\theta) \quad (103)$$

$$(104)$$

10 Phase modulation

For both FLC ON and OFF, phase carries some information too.

Taking phase of (+,+) component, for FLC OFF:

$$
\begin{aligned}
8I(+,+)^{OFF} &= -(s_1 + is_3) \exp i(+\beta\omega x + \alpha\omega y + \gamma\omega) && (105)\\
&= -(s_1 + is_3) \exp i\xi && (106)\\
\Re &= -s_1 \cos\xi + s_3 \sin\xi && (107)\\
\Im &= -s_1 \sin\xi - s_3 \cos\xi && (108)\\
\tan\Phi &= \frac{\Im}{\Re} = \frac{s_1 \sin\xi + s_3 \cos\xi}{s_1 \cos\xi - s_3 \sin\xi} = \frac{\tan\xi + \frac{s_3}{s_1}}{1 - \frac{s_3}{s_1}\tan\xi} && (109)\\
&= \frac{\tan\xi + \frac{\tan 2\chi}{\cos 2\theta}}{1 - \frac{\tan 2\chi}{\cos 2\theta}\tan\xi} && (110)\\
&= \tan(\xi + \Theta) && (111)\\
\Phi(+,+) &= \xi + \Theta && (112)\\
& && (113)
\end{aligned}
$$

where $\tan\Theta = \frac{\tan 2\chi}{\cos 2\theta}$ which is a bit like the electric field amplitude ratio $\frac{|E_y|}{|E_x|} = \frac{\tan 2\chi}{\sin 2\theta}$.

That can in principle give the χ like thing to eliminate from the amplitude measurement to recover θ, but now needs calibration to remove the spectrum dependent offset held in ξ. If the spectrum changes slowly, it might be possible to do the amplitude based χ determination less often by switching the FLC, use that to calibrate or check the phase determination, and then keep the FLC off for several frames.

The three components give:

$$
\begin{aligned}
& && (114)\\
\Phi(+,+) &= \beta\omega x + \gamma\omega + \alpha\omega y + \Theta && (115)\\
\Phi(+,-) &= \beta\omega x + \gamma\omega - \alpha\omega y - \Theta && (116)\\
\Phi(+,0) &= \beta\omega x + \gamma\omega && (117)\\
& && (118)
\end{aligned}
$$

Unfortunately, the desired Θ cannot be separated from ω variations in y. Also $\omega(x,y)$ cannot be obtained directly.

With the FLC on, we get:

$$\begin{aligned}
8I(+,+)^{ON} &= -(s_1 - is_2)\exp i(+\beta\omega x + \gamma\omega + \alpha\omega y) && (119)\\
&= -(s_1 - is_2)\exp i\xi && (120)\\
\Re &= -s_1\cos\xi - s_2\sin\xi && (121)\\
\Im &= -s_1\sin\xi + s_2\cos\xi && (122)\\
\tan\Phi &= \frac{\Im}{\Re} = \frac{s_1\sin\xi - s_2\cos\xi}{s_1\cos\xi + s_2\sin\xi} = \frac{\tan\xi - \frac{s_2}{s_1}}{1 + \frac{s_2}{s_1}\tan\xi} && (123)\\
&= \frac{\tan\xi - \tan 2\theta}{1 + \tan 2\theta\tan\xi} && (124)\\
\Phi(+,+)^{ON} &= \xi - 2\theta && (125)\\
& && (126)\\
\Phi(+,+)^{ON} &= \beta\omega x + \gamma\omega + \alpha\omega y - 2\theta && (127)\\
\Phi(+,-)^{ON} &= \beta\omega x + \gamma\omega - \alpha\omega y + 2\theta && (128)\\
\Phi(+,0)^{ON} &= \beta\omega x + \gamma\omega && (129)\\
& && (130)
\end{aligned}$$

Interlacing the FLC allows us to eliminate ξ:

$$\begin{aligned}
\Sigma\Phi = \frac{\Im\left[I(+,+)^{ON} I(+,+)^{OFF}\right]}{\Re\left[I(+,+)^{ON} I(+,+)^{OFF}\right]} &= \tan(2\xi + 2\theta + \Theta) && (131)\\
\Delta\Phi = \frac{\Im\left[I(+,+)^{ON} / I(+,+)^{OFF}\right]}{\Re\left[I(+,+)^{ON} / I(+,+)^{OFF}\right]} &= \tan(2\theta - \Theta) && (132)\\
& && (133)
\end{aligned}$$

So what actualy is ξ???

$$\begin{aligned}
\xi &= +\beta\omega x + \gamma\omega + \alpha\omega y && (134)\\
&= \omega(\beta x + \alpha y + \gamma) && (135)\\
& && (136)
\end{aligned}$$

For $I(+,0)$ (FLC off):

$$\begin{aligned}
8I(+,0) &= +2s_2\exp i(+\beta\omega x + \gamma\omega) && (137)\\
&= +2s_2\exp i\xi_2 && (138)\\
8I(+,+)^{OFF} &= -(s_1 + is_3)\exp i(+\beta\omega x + \alpha\omega y + \gamma\omega) && (139)\\
&= -(s_1 + is_3)\exp i\xi && (140)\\
\Re &= -s_1\cos\xi + s_3\sin\xi && (141)\\
\Im &= -s_1\sin\xi - s_3\cos\xi && (142)\\
\tan\Phi &= \frac{\Im}{\Re} = \frac{s_1\sin\xi + s_3\cos\xi}{s_1\cos\xi - s_3\sin\xi} = \frac{\tan\xi + \frac{s_3}{s_1}}{1 - \frac{s_3}{s_1}\tan\xi} && (143)\\
&= \frac{\tan\xi + \frac{\tan 2\chi}{\cos 2\theta}}{1 - \frac{\tan 2\chi}{\cos 2\theta}\tan\xi} && (144)\\
&= \tan(\xi + \Phi) && (145)
\end{aligned}$$

11 MSE Geometry without the trig!!

Define vectors:

$\underline{\mathrm{B}}$ Magnetic field.

$\underline{\mathrm{v}}_i$ Velocity vector for beam i.

$\hat{\underline{\mathrm{l}}}$ Unit vector of pixel's line of sight.

$\hat{\underline{\mathrm{u}}}$ Unit vector defining 'up' for a pixel. Perp to $\hat{\underline{\mathrm{l}}}$ but otherwise free.

$\hat{\underline{\mathrm{r}}}$ Defines 'right'. $\hat{\underline{\mathrm{r}}} = \hat{\underline{\mathrm{l}}} \times \hat{\underline{\mathrm{u}}}$.

Lorentz electric field, has direction of π components:

$$\underline{\mathrm{E}} = \underline{\mathrm{v}} \times \underline{\mathrm{B}}. \tag{146}$$

We define the raw polarisation angle θ as the angle from the LOS up $\hat{\underline{\mathrm{u}}}$:

$$\tan\theta = \frac{(\underline{\mathrm{v}} \times \underline{\mathrm{B}}) \cdot \hat{\underline{\mathrm{r}}}}{(\underline{\mathrm{v}} \times \underline{\mathrm{B}}) \cdot \hat{\underline{\mathrm{u}}}} \tag{147}$$

Break $\underline{\mathrm{B}}$ into the global Tokamak polar coordinates $(\hat{\underline{\mathrm{R}}}, \hat{\underline{\phi}}, \hat{\underline{\mathrm{Z}}})$:

$$\begin{aligned}
\underline{\mathrm{B}} &= B_R\hat{\underline{\mathrm{R}}} + B_Z\hat{\underline{\mathrm{Z}}} + B_\phi\hat{\underline{\phi}} && (148)\\
\underline{\mathrm{v}} \times \underline{\mathrm{B}} &= (\underline{\mathrm{v}} \times \hat{\underline{\mathrm{R}}})B_R + (\underline{\mathrm{v}} \times \hat{\underline{\mathrm{Z}}})B_Z + (\underline{\mathrm{v}} \times \hat{\underline{\phi}})B_\phi && (149)\\
\tan\theta &= \frac{\left(\underline{\mathrm{v}} \times \hat{\underline{\phi}} \cdot \hat{\underline{\mathrm{r}}}\right) B_\phi + \left(\underline{\mathrm{v}} \times \hat{\underline{\mathrm{R}}} \cdot \hat{\underline{\mathrm{r}}}\right) B_R + \left(\underline{\mathrm{v}} \times \hat{\underline{\mathrm{Z}}} \cdot \hat{\underline{\mathrm{r}}}\right) B_Z}{\left(\underline{\mathrm{v}} \times \hat{\underline{\phi}} \cdot \hat{\underline{\mathrm{u}}}\right) B_\phi + \left(\underline{\mathrm{v}} \times \hat{\underline{\mathrm{R}}} \cdot \hat{\underline{\mathrm{u}}}\right) B_R + \left(\underline{\mathrm{v}} \times \hat{\underline{\mathrm{Z}}} \cdot \hat{\underline{\mathrm{u}}}\right) B_Z} && (150)\\
& && (151)
\end{aligned}$$

Divide through by $\left(\underline{\mathrm{v}} \times \hat{\underline{\phi}} \cdot \hat{\underline{\mathrm{u}}}\right)$:

$$\begin{aligned}
\tan\theta &= \left[\frac{(\underline{\mathrm{v}} \times \hat{\underline{\phi}} \cdot \hat{\underline{\mathrm{r}}})}{(\underline{\mathrm{v}} \times \hat{\underline{\phi}} \cdot \hat{\underline{\mathrm{u}}})} + \frac{(\underline{\mathrm{v}} \times \hat{\underline{\mathrm{R}}} \cdot \hat{\underline{\mathrm{r}}})}{(\underline{\mathrm{v}} \times \hat{\underline{\phi}} \cdot \hat{\underline{\mathrm{u}}})}\frac{B_R}{B_\phi} + \frac{(\underline{\mathrm{v}} \times \hat{\underline{\mathrm{Z}}} \cdot \hat{\underline{\mathrm{r}}})}{(\underline{\mathrm{v}} \times \hat{\underline{\phi}} \cdot \hat{\underline{\mathrm{u}}})}\frac{B_Z}{B_\phi}\right] && (152)\\
&\quad \times \left[1 + \frac{(\underline{\mathrm{v}} \times \hat{\underline{\mathrm{R}}} \cdot \hat{\underline{\mathrm{u}}})}{(\underline{\mathrm{v}} \times \hat{\underline{\phi}} \cdot \hat{\underline{\mathrm{u}}})}\frac{B_R}{B_\phi} + \frac{(\underline{\mathrm{v}} \times \hat{\underline{\mathrm{Z}}} \cdot \hat{\underline{\mathrm{u}}})}{(\underline{\mathrm{v}} \times \hat{\underline{\phi}} \cdot \hat{\underline{\mathrm{u}}})}\frac{B_Z}{B_\phi}\right]^{-1} && (153)
\end{aligned}$$

For normal plasmas, we can assume $\frac{B_R}{B_\phi} < \frac{B_Z}{B_\phi} << 1$.

Binomial expand and drop terms in $(\frac{B_{R/Z}}{B_\phi})^2$:

$$\begin{aligned}
\tan\theta &= \frac{(\underline{\mathrm{v}} \times \hat{\underline{\phi}} \cdot \hat{\underline{\mathrm{r}}})}{(\underline{\mathrm{v}} \times \hat{\underline{\phi}} \cdot \hat{\underline{\mathrm{u}}})} && (154)\\
&\quad + \left[\frac{(\underline{\mathrm{v}} \times \hat{\underline{\mathrm{R}}} \cdot \hat{\underline{\mathrm{r}}})}{(\underline{\mathrm{v}} \times \hat{\underline{\phi}} \cdot \hat{\underline{\mathrm{u}}})} - \frac{(\underline{\mathrm{v}} \times \hat{\underline{\phi}} \cdot \hat{\underline{\mathrm{r}}})}{(\underline{\mathrm{v}} \times \hat{\underline{\phi}} \cdot \hat{\underline{\mathrm{u}}})}\frac{(\underline{\mathrm{v}} \times \hat{\underline{\mathrm{R}}} \cdot \hat{\underline{\mathrm{u}}})}{(\underline{\mathrm{v}} \times \hat{\underline{\phi}} \cdot \hat{\underline{\mathrm{u}}})}\right]\frac{B_R}{B_\phi} && (155)\\
&\quad + \left[\frac{(\underline{\mathrm{v}} \times \hat{\underline{\mathrm{Z}}} \cdot \hat{\underline{\mathrm{r}}})}{(\underline{\mathrm{v}} \times \hat{\underline{\phi}} \cdot \hat{\underline{\mathrm{u}}})} - \frac{(\underline{\mathrm{v}} \times \hat{\underline{\phi}} \cdot \hat{\underline{\mathrm{r}}})}{(\underline{\mathrm{v}} \times \hat{\underline{\phi}} \cdot \hat{\underline{\mathrm{u}}})}\frac{(\underline{\mathrm{v}} \times \hat{\underline{\mathrm{Z}}} \cdot \hat{\underline{\mathrm{u}}})}{(\underline{\mathrm{v}} \times \hat{\underline{\phi}} \cdot \hat{\underline{\mathrm{u}}})}\right]\frac{B_Z}{B_\phi} && (156)\\
&= a\frac{B_R}{B_\phi} + b\frac{B_Z}{B_\phi} + c && (157)
\end{aligned}$$

With a choice of $\underline{\hat{u}}$ that gets the image average as close as possible for beam Q3, AUG modelling (using measured camera view) gives, on average:

$$\tan\theta_1 \approx 0.024\frac{B_R}{B_\phi} + 0.408\frac{B_Z}{B_\phi} + 0.179 \quad (158)$$

$$\tan\theta_2 \approx 0.057\frac{B_R}{B_\phi} + 0.605\frac{B_Z}{B_\phi} + 0.193 \quad (159)$$

$$\tan\theta_3 \approx 0.055\frac{B_R}{B_\phi} + 0.584\frac{B_Z}{B_\phi} - 0.001 \quad (160)$$

$$\tan\theta_4 \approx 0.037\frac{B_R}{B_\phi} + 0.399\frac{B_Z}{B_\phi} - 0.003 \quad (161)$$

$$(162)$$

The measured angle has some unknown offset from any arbitrary choice of $\underline{\hat{u}}$: $\theta' = \theta + \theta_0$. If we have measurements with two beams, the differece removes that offset. With the same choice of $\underline{\hat{u}}$, so that $c_3 \approx 0$:

$$\tan(\theta_i' - \theta_3') = \tan(\theta_i - \theta_3) = \quad (163)$$

$$= \frac{(a_i - a_3)\frac{B_R}{B_\phi} + (b_i - b_3)\frac{B_Z}{B_\phi} + (c_i - c_3)}{1 + (a_i\frac{B_R}{B_\phi} + b_i\frac{B_Z}{B_\phi} + c_i)(a_3\frac{B_R}{B_\phi} + b_3\frac{B_Z}{B_\phi} + c_3)} \quad (164)$$

$$\approx \frac{(a_i - a_3)\frac{B_R}{B_\phi} + (b_i - b_3)\frac{B_Z}{B_\phi} + c_i}{1 + c_i a_3\frac{B_R}{B_\phi} + c_i b_3\frac{B_Z}{B_\phi}} \quad (165)$$

$$\approx (a_i - a_3 - c_i^2 a_3)\frac{B_R}{B_\phi} + (b_i - b_3 + c_i^2 b_3)\frac{B_Z}{B_\phi} + c_i \quad (166)$$

$$(167)$$

$$\tan(\theta_1' - \theta_3') \approx 0.179 \quad -0.033\frac{B_R}{B_\phi} \quad -0.195\frac{B_Z}{B_\phi} \quad (168)$$

$$\tan(\theta_2' - \theta_3') \approx 0.193 \quad +0.001\frac{B_Z}{B_\phi} \quad (169)$$

$$\tan(\theta_4' - \theta_3') \approx -0.003 \quad -0.018\frac{B_R}{B_\phi} \quad -0.185\frac{B_Z}{B_\phi} \quad (170)$$

$$(171)$$

Which appears to suggest that beams 2 and 3 should always have the same difference - 11^o, independent of the magnetic field, choice of $\underline{\hat{u}}$, or instrumental offsets. This is reasonable, since they have roughly the same but opposite inclination to the horizontal plane, with all the other vectors roughly the same. However, you'd expect the difference to be 8.2^o - twice the inclination angle.

The difference between beams 4 and 3, both of which are nicely in view, given a more direct measurement of $\frac{B_Z}{B_\phi}$, independent of instrumental offsets, and without any signifcant raw difference between the angle - which reduces errors due to the non-linear intrinsic contrast calibration (μ). In principle, separating $\frac{B_R}{B_\phi}$ might even be possible, but is probably well beyond the sensitivity available.

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